The Foundations of Finance in Game-Theoretic Probability

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September 16, 2010

Abstract

This paper develops Vovk's notion of game-theoretic quadratic variation. In particular, it considers different constructions of quadratic variation and the connection between them, as well as relative quadratic variation and its linkage to the absolute quadratic variation. No probability assumptions are made.

1 Introduction

Continuous-time finance is usually studied in the measure-theoretic framework. Vovk and Shafer ([2], [10], [9]) studied it in the game-theoretical setting using non-standard analysis, but this was only partly successful. The theory without non-standard analysis was developed by Vovk in [4] and [7] assuming that quadratic variation is imposed on the game.

The Japanese school of game-theoretic probability proposed an alternative framework for continuous time stochastic processes in [3] in which quadratic variation emerges from market efficiency. This approach has been further developed by Vovk ([5], [6] and [8]). This study develops their approach in a way that can be used in mathematical finance.

1.1 Game-theoretic probability in finance in discrete time

In their 2001 book, Shafer and Vovk study games for pricing European and American options as well as for diffusion processes in discrete time. Each game is a *perfect-information game*, in which every move of every player is immediately revealed to other players. The players in these games are Market, Investor, Skeptic, Reality, Forecaster and Speculator.

Market is believed to be *efficient* to certain degree, in the sense, that Speculator is unlikely to multiply his capital by a large factor. This idea is used to assign game-theoretic probabilities to events. For example, an event is said to happen *almost surely* if Speculator has a strategy that multiplies his capital by an infinite factor if the event fails. Equipped with this notion of contingency, Shafer and Vovk provide their own versions of established results in finance. The major advantage of this approach is that it does not assume any probability measure, which is often considered to be unrealistic.

In [10] (published as [11]) Vovk and Shafer study the game-theoretical *capital asset price model* (CAPM) in discrete time. In particular, they consider the following game

Basic Capital Asset Pricing Game (Basic CAPG)

Players: Investor, Market, Speculator

Parameters:

Natural number K (number of non-index securities in the market)
Natural number N (number of rounds or trading period)
Real number α satisfying $0 < \alpha \leq 1$ (significance level)

Protocol:

$\mathcal{G}_0 := 1$.
$\mathcal{H}_0 := 1$.
$\mathcal{M}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
Investor selects $g_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} g_n^k = 1$.
Speculator selects $h_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} h_n^k = 1$.
Market selects $x_n \in (-1, \infty)^{K+1}$.
$\mathcal{G}_n := \mathcal{G}_{n-1} \sum_{k=0}^{K} g_n^k (1 + x_n^k)$.
$\mathcal{H}_n := \mathcal{G}_{n-1} \sum_{k=0}^{K} h_n^k (1 + x_n^k)$.
$\mathcal{M}_n := \mathcal{M}_{n-1}(1 + x_n^0)$.

Here x_n^0 is the return on market index at round n, x_n^k is the return on security k in round n, $\mathcal{M}_n$ is the capital at the end of round n resulting from investing capital of size one in market at the beginning of the game, $\mathcal{G}_n$ is the capital of Investor at the end of round n and $\mathcal{H}_n$ is the capital of Speculator at the end of round n. The only remaining notions are

$$s_n := \frac{\mathcal{G}_n - \mathcal{G}_{n-1}}{\mathcal{G}_{n-1}}, \qquad m_n := x_n^0.$$

Let

$$\mu_s := \frac{1}{T} \sum_{n=1}^{N} s_n, \qquad \mu_m := \frac{1}{T} \sum_{n=1}^{N} m_n, \tag{1}$$

$$\sigma_m^2 := \frac{1}{T} \sum_{n=1}^{N} s_n^2, \qquad \sigma_{sm} := \frac{1}{T} \sum_{n=1}^{N} s_n m_n, \tag{2}$$

and let A be the event that μ_s approximates $\mu_m - \sigma_m^2 + \sigma_{sm}$. Vovk and Shafer showed that A happens almost surely. Moreover, the gave explicit bounds to justify this approximation. Also, it is interesting to note that the quantities defined in (1) are of empirical nature.

1.2 The accomplishments using non-standard analysis

Non-standard analysis operates with such notions as *infinitely large* and *infinitesimal*, which intuitively mean "very large" and "very small". To generalize the results of game-theoretic probability in finance in discrete time, Shafer and Vovk divide an interval $[0, T]$ into an infinitely large number N of steps of equal infinitesimal length $dt := T/N$. They consider a game of N rounds with the amount of time between successive rounds equal to dt. By the *transfer principle* some non-standard theorems can be deduced directly from the corresponding standard theorems. Therefore, the results of the discrete framework readily generalize to their continuous time counterparts.

Shafer and Vovk in [2] consider games for pricing options and for diffusion processes. Among other results, a variant of the Black-Scholes formula is deduced.

In [9] Vovk and Shafer used the same setting with non-standard analysis in the following game, which is a continuous version of the game described in §1.1

Basic Capital Asset Pricing Protocol

Players: Investor, Market, Speculator

Parameters:

Natural number K (number of non-index securities in the market)
Infinite natural number N (number of rounds or trading period)

Protocol:

$\mathcal{G}_0 := 1$.
$\mathcal{H}_0 := 1$.
$\mathcal{M}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
Investor selects $g_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} g_n^k = 1$.
Speculator selects $h_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^{K} h_n^k = 1$.
Market selects $x_n \in (-1, \infty)^{K+1}$.
$\mathcal{G}_n := \mathcal{G}_{n-1} \sum_{k=0}^{K} g_n^k (1 + x_n^k)$.
$\mathcal{H}_n := \mathcal{G}_{n-1} \sum_{k=0}^{K} h_n^k (1 + x_n^k)$.
$\mathcal{M}_n := \mathcal{M}_{n-1} (1 + x_n^0)$.

Restrictions:

Market and investor are required to make σ_s^2 and σ_m^2 finite and to make $\max_n |s_n|$ and $\max_n |m_n|$ infinitesimal.

For this game it was proven that for any $\epsilon > 0$

$$|\mu_s - \mu_m + \sigma_s^2 - \sigma_{sm}| < \epsilon(1 + \sigma_{s-m}^2) \tag{3}$$

almost surely, where

$$\sigma_{s-m}^2 := \frac{1}{T}\sum_{n=1}^{N}(s_n - m_n)^2.$$

This is a game-theoretic continuous analogue of the CAPM in the standard financial theory. The $\sigma_{s-m}^2/2$ is called a *theoretical performance deficit* and the following result for it was obtained: for any $\epsilon > 0$

$$|\lambda_s - \lambda_m + \frac{1}{2}\sigma_{s-m}^2| < \epsilon(1 + \sigma_{s-m}^2) \tag{4}$$

almost surely.

Although non-standard analysis seems to be a natural choice in the game-theoretical framework, one may find it awkward and, thus, unsatisfactory since its machinery highly depends on the selection of the *ultrafilter*, a special family of subsets of $\mathbb{N}$. It cannot be chosen uniquely and there is no constructive version of it.

1.3 Vovk's recent work on continuous time stochastic processes

Inspired by [3], Vovk in [5], [6] and [8] develops the theory of continuous time processes in the game-theoretic probability framework without non-standard analysis. This theory relies only on a trading protocol.

Though the formal picture will be given later in Section 2.1, it would be beneficial to provide some insight into the idea behind it. We consider a game, where Reality (market) produces some continuous function $\omega : [0, \infty) \to \mathbb{R}$. For each time $t \in [0, \infty)$, the value of $\omega(t)$ represents the price of the financial asset at that time. Skeptic chooses a trading strategy prior to Reality's choice of ω. Skeptic's trading strategy is a countable sum of elementary trading strategies. An elementary trading strategy is an initial capital together with a rule which says how much to bet at specific times, which depend on the available information about ω. As shown by Vovk, this framework gives enough flexibility to define upper and lower probabilities and to make assessments about continuous time processes.

In [5] it was shown that ω has certain properties of Brownian motion including the absence of isolated zeros and the absence of points of strict increase or decrease. The main result of [6] is that almost surely

$$\sup_{\kappa}\sum_{i=1}^{n}|\omega(t_i) - \omega(t_{i-1})|^2,$$

is finite (here n ranges over all positive integers and κ over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$), and that the analogous quantity when the exponent 2 is replaced by a smaller number is not finite, when ω is constant.

The further development of the theory in [8] shows that the Wiener measure, a measure generated by Brownian motion, emerges naturally in this setting. Its main result already contains the results from [5] and [6] and is presented in two ways, abstract and constructive, which provides different perspectives on the matter.

1.4 The need to develop quadratic variation in Vovk's framework.

Although quadratic variation was studied in [8], we still need to develop it further. It is exactly the objective of this paper. In the following sections we review Vovk's constructive version of quadratic variation, discuss its construction in both Lebesque and Riemannian sense, prove the existence of relative quadratic variation and show how it is connected to its absolute counterpart. These are the necessary steps in the further study of the capital asset price model in Vovk's game-theoretical probability framework in continuous time.

Sections 2 and 3 basically repeats some results of Vovk's 2009 paper [8]. Sections 4 and 5 contain the actual results of this work and Section 6 concludes it.

2 Game setting, upper probabilities and game-theoretical version Hoeffding's inequality

In this section we describe the game, define process, event, trading strategy, upper and lower probabilities, and also present a useful version of Hoeffding's inequality as developed by Vovk in [8].

2.1 Vovk's framework

We consider a game between two players, Reality (a financial market) and Skeptic (a trader), over the time interval $[0, \infty)$. First Skeptic chooses his trading strategy and then Reality chooses a continuous function $\omega : [0, \infty) \to \mathbb{R}$ (the price process of a security).

We define Ω to be the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$. For each $t \in [0, \infty)$, let $\mathcal{F}_t$ be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable. A *process* $\mathfrak{S}$ is a family of functions $\mathfrak{S}_t : \Omega \to \mathbb{R}$, $t \in [0, \infty)$, each $\mathfrak{S}_t$ being $\mathcal{F}_t$-measurable; its *sample paths* are the functions $t \mapsto \mathfrak{S}_t(\omega)$. An *event* is an element of the σ-algebra $\mathcal{F}_\infty := \vee_t \mathcal{F}_t$ (also denoted by $\mathcal{F}$). Stopping times $\tau : \Omega \to [0, \infty]$ w.r. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $\mathfrak{S}_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $\mathfrak{S}_\tau(\omega)$, respectively (occasionally, the argument ω will be omitted in other cases as well).

An *elementary trading strategy* G consists of an increasing sequence of stopping times $0 \le \tau_1 \le \tau_2 \le \dots$ such that $\lim_{n\to\infty} \tau_n(\omega) = \infty$ for each $\omega \in \Omega$, and, of a sequence of bounded $\mathcal{F}_{\tau_n}$-measurable functions h_n, $n = 1, 2, \ldots$. To such a G and an *initial capital* $c \in \mathbb{R}$ corresponds the *elementary capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big), \quad t \in [0, \infty), \tag{5}$$

Note, that the value of the sum is finite for each t. The value $h_n(\omega)$ will be called Skeptic's *bet* (or *stake*) at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will be called Skeptic's capital at time t.

A *positive continuous capital process* is any process $\mathfrak{S}$ that can be represented in the form

$$\mathfrak{S}_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n, c_n}(\omega), \tag{6}$$

where the elementary capital processes are $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be non-negative, for all t and ω, and the positive series $\sum_{n=1}^{\infty} c_n$ converges. The sum (6) is always non-negative and allowed to be infinite. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $\mathfrak{S}_0(\omega) = \sum_{n=1}^{\infty} c_n$ also does not depend on ω and will sometimes be abbreviated to $\mathfrak{S}_0$. Let $\mathcal{S}$ be the set of all positive continuous capital processes on Ω. Sometimes we will refer to a positive continuous capital process simply as a trading strategy.

Any real-valued function on Ω is called a *variable.* We define an *upper price* of a bounded variable ξ as

$$\overline{\mathbb{E}}(\xi) := \inf\big\{\mathfrak{S}_0 \bigm| \mathfrak{S} \in \mathcal{S} : \forall \omega \in \Omega \ \liminf_{t \to \infty} \mathfrak{S}_t(\omega) \geq \xi(\omega)\big\}. \tag{7}$$

Therefore, an upper price of ξ is the least starting capital (here we disregard the infimum) needed to achieve at least $\xi(\omega)$ for any $\omega \in \Omega$.

For any set $E \subseteq \Omega$ the upper price of the indicator function of E, $\mathbf{1}_E$, will be called the *upper probability* of E and will be denoted as $\overline{\mathbb{P}}(E)$. That is

$$\overline{\mathbb{P}}(E) := \inf\big\{\mathfrak{S}_0 \bigm| \mathfrak{S} \in \mathcal{S} : \forall \omega \in \Omega \ \liminf_{t \to \infty} \mathfrak{S}_t(\omega) \geq \mathbf{1}_E(\omega)\big\}, \tag{8}$$

Intuitively, an event with lower probability is one that you can bet on at lower cost - i.e., for which you put up less money initially in order to get one dollar if the event happens.

We say that $E \subseteq \Omega$ is *null* if $\overline{\mathbb{P}}(E) = 0$. Correspondingly, a set $E \subseteq \Omega$ is *almost certain* or *almost sure* if $\overline{\mathbb{P}}(E^c) = 0$, where $E^c := \Omega \setminus E$ stands for the complement of E. A property of $\omega \in \Omega$ will be said to hold *almost surely* or *almost surely*, or for *almost all* ω, if the set of ω where it fails is null.

The *lower probability* is defined as:

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c).$$

2.2 Hoeffding's inequality

In this section we provide an important result by Vovk from [8]. This is a game-theoretic version of Hoeffding's inequality (the original version is in [1]). It will be useful in the later proof of existence of quadratic variation.

GAME OF FORECASTING BOUNDED VARIABLES

Players: Skeptic, Forecaster, Reality

Protocol:

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots$:
Forecaster announces interval $[a_n, b_n] \subseteq \mathbb{R}$
and number $\mu_n \in (a_n, b_n)$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [a_n, b_n]$.
Skeptic announces $\mathcal{K}_n \leq \mathcal{K}_{n-1} + M_n(x_n - \mu_n)$.

This is a perfect-information game: each player can see the other players' moves before making his or hers. Each round n of the game begins with Forecaster outputting an interval $[a_n, b_n]$ which, he believes, will capture the actual observation x_n chosen by Reality. He also gives a guess μ_n for x_n. We can regard μ_n as the price of a ticket which pays x_n after Reality's move becomes known. Skeptic can purchase any number M_n (including non-positive) of such tickets. Skeptic can choose his initial capital $\mathcal{K}_0$ and is allowed to throw away part of his money at the end of each round.

We call any real-valued function defined on all finite sequences $(a_1, b_1, \mu_1, x_1, \ldots, a_N, b_N, \mu_N, x_N)$, $N = 0, 1, \ldots$, of Forecaster's and Reality's moves a *process* . Fixing a strategy for Skeptic, will make Skeptic's capital $\mathcal{K}_N$, $N = 0, 1, \ldots$, a function of Forecaster's and Reality's previous moves; in other words, Skeptic's capital becomes a process. The processes that can be obtained this way are called *discrete supercapital processes.*

Theorem 1. *For any $h \in \mathbb{R}$, the process*

$$\prod_{n=1}^{N} \exp\left(h(x_n - \mu_n) - \frac{h^2}{8}(b_n - a_n)^2\right)$$

is a supercapital process.

3 Quadratic variation.

In this section we study quadratic variation. The proof of Lemma 1 is a slightly rewritten version of Vovk's original proof from [8].

For $n = 0, 1, 2...$, let $\mathbb{D}_n := \{l2^{-n} \mid l \in \mathbb{Z}\}$ and define a sequence of stopping times T^n_k, $k = -1, 0, 1, 2, \ldots$, inductively by $T^n_{-1} := 0$,

$$T^n_0(\omega) := \inf \{t \geq 0 \mid \omega(t) \in \mathbb{D}_n\},$$
$$T^n_k(\omega) := \inf \left\{t \geq T^n_{k-1} \mid \omega(t) \in \mathbb{D}_n \;\&\; \omega(t) \neq \omega(T^n_{k-1})\right\}, \quad k = 1, 2, \ldots$$

where $\inf \emptyset := \infty$.

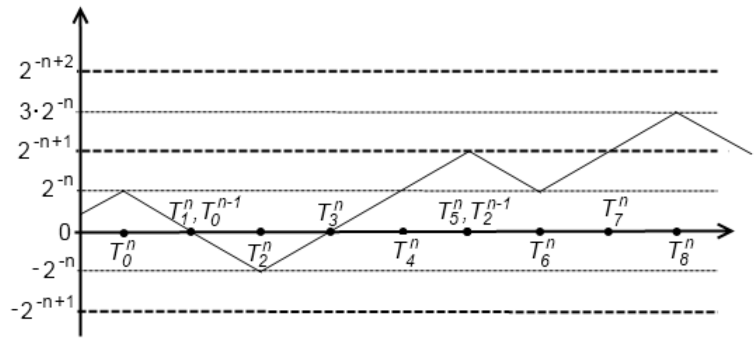


Figure 1: Stopping times for a piecewise linear function ω.

For each $t \in [0, \infty)$ and $\omega \in \Omega$, define

$$A^n_t(\omega) := \sum_{k=0}^{\infty} \left(\omega(T^n_k \wedge t) - \omega(T^n_{k-1} \wedge t)\right)^2, \quad n = 0, 1, 2, \ldots. \tag{9}$$

Lemma 1. *For each $T > 0$, it is almost certain that $t \in [0, T] \mapsto A^n_t$ is a Cauchy sequence of functions in $C[0, T]$.*

Proof. Fix $T > 0$ and fix temporarily $n \in \{1, 2, \ldots\}$. Let $\kappa \in \{0, 1\}$ be such that $T^{n-1}_0 = T^n_\kappa$, i.e. it is the number of times A^n_t increased on the interval $[0, T^{n-1}_0)$. Note, that for any $k = 1, 2, \ldots$

$$\omega(T^n_k) - \omega(T^n_{k-1}) := \begin{cases} 2^{-n} & \text{if } \omega(T^n_k) > \omega(T^n_{k-1}) \\ -2^{-n} & \text{otherwise,} \end{cases}$$

For each $k = 1, 2, \ldots$, let

$$\xi_k := \begin{cases} 1 & \text{if } \omega(T^n_{\kappa+2k}) = \omega(T^n_{\kappa+2k-2}) \\ -1 & \text{otherwise.} \end{cases}$$

If ω were generated by Brownian motion, ξ_k would be a random variable taking value j, $j \in \{1, -1\}$, with probability $1/2$ and its expected value would be 0. This remains true in our setting in the sense that the game-theoretical expected value of ξ_k at time $T^n_{\kappa+2k-2}$ is 0. To see this consider the following elementary trading strategy that, when started with initial capital 0 at time $T^n_{\kappa+2k-2}$, ends with ξ_k at time $T^n_{\kappa+2k}$ (given both times are finite): at time $T^n_{\kappa+2k-1}$ bet -2^n if $\omega(T^n_{\kappa+2k-1}) > \omega(T^n_{\kappa+2k-2})$ and bet 2^n otherwise. That is

$$\xi_k = \sum_{r=\kappa+2k-2}^{\kappa+2k-1} h_r(\omega)\left(\omega(T^n_{r+1}) - \omega(T^n_r)\right), \tag{10}$$

where $h_{\kappa+2k-2}(\omega) = 0$ and

$$h_{\kappa+2k-1}(\omega) := \begin{cases} -2^n & \text{if } \omega(T^n_{\kappa+2k-1}) > \omega(T^n_{\kappa+2k-2}) \\ 2^n & \text{otherwise.} \end{cases}$$

Let η_k denote the increment of the process $A^n_t - A^{n-1}_t$ over the time interval $[T^n_{\kappa+2k-2}, T^n_{\kappa+2k}]$. That is

$$\begin{aligned} \eta_k &:= A^n_{T^n_{\kappa+2k}} - A^{n-1}_{T^n_{\kappa+2k}} - \left(A^n_{T^n_{\kappa+2k-2}} - A^{n-1}_{T^n_{\kappa+2k-2}}\right) \\ &= A^n_{T^n_{\kappa+2k}} - A^n_{T^n_{\kappa+2k-2}} - \left(A^{n-1}_{T^n_{\kappa+2k}} - A^{n-1}_{T^n_{\kappa+2k-2}}\right). \end{aligned}$$

We want to show that $\eta_k \in \{-2^{-2n+1}, 2^{-2n+1}\}$ and its game-theoretical expected value at time $T^n_{\kappa+2k-2}$ is equal to zero.

Since

$$\begin{aligned} A^n_{T^n_{\kappa+2k}} - A^n_{T^n_{\kappa+2k-2}} &= \left(\omega(T^n_{\kappa+2k-1}) - \omega(T^n_{\kappa+2k-2})\right)^2 \\ &\quad + \left(\omega(T^n_{\kappa+2k-1}) - \omega(T^n_{\kappa+2k-2})\right)^2 \\ &\qquad\qquad = (2^{-n})^2 + (2^{-n})^2 = 2^{-2n+1}. \end{aligned}$$

and

$$\begin{aligned} A^{n-1}_{T^n_{\kappa+2k}} - A^{n-1}_{T^n_{\kappa+2k-2}} &= \left(\omega(T^n_{\kappa+2k}) - \omega(T^n_{\kappa+2k-2})\right)^2 \mathbf{1}_{\{\xi_k=-1\}} \\ &= 2^{-2n+2}\mathbf{1}_{\{\xi_k=-1\}}. \end{aligned}$$

we have

$$\eta_k = 2^{-2n+1} - 2^{-2n+2}\mathbf{1}_{\{\xi_k=-1\}} = 2^{-2n+1}\xi_k.$$

Now, we can apply the game-theoretic version of Hoeffding's inequality to η_k: for any constant $\lambda \in \mathbb{R}$ there exists a positive supercapital process $\mathfrak{S}$ with $\mathfrak{S}_0 = 1$ such that, for all $K = 0, 1, 2, \ldots$,

$$\mathfrak{S}_{T^n_{\kappa+2K}} = \prod_{k=1}^{K} \exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right). \tag{11}$$

The Hoeffding's inequality guarantees that there is a capital process $\mathfrak{S}$, which is positive at $T^n_{\kappa+2K}$, $K = 0, 1, \ldots$, but we also want to make sure that it is positive for all $t \in (T^n_{\kappa+2k+2}, T^n_{\kappa+2k})$ and for all $k = 1, 2, \ldots, K$. The following expression holds and follows directly from the proof of Hoeffding's inequality in [8]

$$\exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right) - 1 \le \eta_k \frac{e^{\lambda 2^{-2n+1}} - e^{\lambda - 2^{-2n+1}}}{2^{-2n+2}} \exp\left(-2^{-4n+1}\lambda^2\right). \quad (12)$$

We can rewrite it in terms of ξ

$$\exp\left(\lambda 2^{-2n+1}\xi_k - 2^{-4n+1}\lambda^2\right) - 1 \le \xi_k \frac{e^{\lambda 2^{-2n+1}} - e^{\lambda - 2^{-2n+1}}}{2} \exp\left(-2^{-4n+1}\lambda^2\right).$$

Furthermore, recalling (10) we obtain

$$\exp\left(\lambda 2^{-2n+1} \sum_{r=\kappa+2k-2}^{\kappa+2k-1} h_r(\omega)\left(\omega(T^n_{r+1}) - \omega(T^n_r)\right) - 2^{-4n+1}\lambda^2\right) - 1$$
$$\le \frac{e^{\lambda 2^{-2n+1}} - e^{\lambda - 2^{-2n+1}}}{2} \exp\left(-2^{-4n+1}\lambda^2\right) \sum_{r=\kappa+2k-2}^{\kappa+2k-1} h_r(\omega)\left(\omega(T^n_{r+1}) - \omega(T^n_r)\right). \quad (13)$$

Denote

$$H_r(\omega) := \frac{e^{\lambda 2^{-2n+1}} - e^{\lambda - 2^{-2n+1}}}{2} \exp\left(-2^{-4n+1}\lambda^2\right) h_r(\omega).$$

Plugging it into (13) yields

$$\exp\left(\lambda 2^{-2n+1} \sum_{r=\kappa+2k-2}^{\kappa+2k-1} h_r(\omega)\left(\omega(T^n_{r+1}) - \omega(T^n_r)\right) - 2^{-4n+1}\lambda^2\right) - 1$$
$$\le \sum_{r=\kappa+2k-2}^{\kappa+2k-1} H_r(\omega)\left(\omega(T^n_{r+1}) - \omega(T^n_r)\right). \quad (14)$$

Both sides of (14) are equal to the corresponding sides of (12). Thus, a strategy defined by betting $H_{\kappa+2k-2}(\omega)$ (which is equal to zero) at time $T^n_{\kappa+2k-2}$ and $H_{\kappa+2k-1}(\omega)$ at time $T^n_{\kappa+2k-1}$ will produce at least $\exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right)$ at time $T^n_{\kappa+2k}$ given that the initial capital at time $T^n_{\kappa+2k-2}$ is of size 1.

For all $t \in [T^n_{\kappa+2k-2}, T^n_{\kappa+2k}]$ define.

$$\delta_k(t) := 2^{-2n+1} \sum_{r=\kappa+2k-2}^{\kappa+2k-1} h_r(\omega)\left(\omega(T^n_{r+1} \wedge t) - \omega(T^n_r \wedge t)\right)$$

Since ω is continuous, $\delta_k(t) \in [-2^{-2n+1}, 2^{-2n+1}]$. Therefore, the inequality (12) with $\delta_k(t)$ in place of η_k is true. A consequence of this is the following inequality (an analogue of (14))

$$\exp\left(\lambda 2^{-2n+1}\delta_k(t) - 2^{-4n+1}\lambda^2\right) - 1$$
$$\le \sum_{r=\kappa+2k-2}^{\kappa+2k-1} H_r(\omega)\left(\omega(T^n_{r+1} \wedge t) - \omega(T^n_r \wedge t)\right), \quad (15)$$

which shows that a capital process $\mathfrak{S}$ defined by strategy discussed above will be positive at any $t \in [T^n_{\kappa+2k-2}, T^n_{\kappa+2k}]$ for all $k = 1, 2, \ldots, K$.

The sum of (11) over $n = 1, 2, \ldots$ with weights $2^{-n}\alpha/2$, $\alpha > 0$, will also be a positive supercapital process, and with lower probability at least $1 - \alpha/2$ it will not exceed $2^n 2/\alpha$. Therefore, none of these processes in this sum will ever exceed $2^n 2/\alpha$. The inequality

$$\prod_{k=1}^{K} \exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right) \le 2^n \frac{2}{\alpha} \le e^n \frac{2}{\alpha}$$

can be equivalently rewritten as

$$\lambda \sum_{k=1}^{K} \eta_k \le K\lambda^2 2^{-4n+1} + n + \ln\frac{2}{\alpha}. \tag{16}$$

Plugging in the identities

$$K = \frac{A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}}{2^{-2n+1}}, \tag{17}$$

$$\sum_{k=1}^{K} \eta_k = \left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right),$$

and taking $\lambda := 2^n$, we can transform (16) to

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \le 2^{-n}\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) + \frac{n + \ln\frac{2}{\alpha}}{2^n}, \tag{18}$$

which implies

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \le 2^{-n} A^n_{T^n_{\kappa+2K}} + 2^{-2n+1} + \frac{n + \ln\frac{2}{\alpha}}{2^n}. \tag{19}$$

This is true for any $K = 0, 1, 2, \ldots$; choosing the largest K such that $T^n_{\kappa+2K} \le t$, we obtain

$$A^n_t - A^{n-1}_t \le 2^{-n} A^n_t + 2^{-2n+2} + \frac{n + \ln\frac{2}{\alpha}}{2^n}, \tag{20}$$

for any $t \in [0, \infty)$. The case $t < T^n_\kappa$ is considered separately by finding the maximum $n^* \in \mathbb{N}$ such that $t > T^{n*-1}_0$ and applying the same reasoning for $A^{n^*}_t - A^{n^*-1}_t$; if there is no such n^*, then ω is constant and $A^n_t = A^{n-1}_t = 0$ for all $n \in \mathbb{N}$ and all $t \in [0, \infty)$.

Proceeding in the same way but taking $\lambda := -2^n$, we obtain

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \ge -2^{-n}\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \frac{n + \ln\frac{2}{\alpha}}{2^n}$$

instead of (18) and

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \ge -2^{-n} A^n_{T^n_{\kappa+2K}} - 2^{-2n+1} - \frac{n + \ln\frac{2}{\alpha}}{2^n}$$

instead of (19), which gives

$$A_t^n - A_t^{n-1} \geq -2^{-n} A_t^n - 2^{-2n+2} - \frac{n + \ln \frac{2}{\alpha}}{2^n} \tag{21}$$

instead of (20). The consequence of the inequalities (20) and (21) is

$$|A_t^n - A_t^{n-1}| \leq 2^{-n} A_t^n + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n}, \tag{22}$$

which holds with lower probability at least $1 - \alpha$.

To prove the statement of the lemma it remains to show that the sequence $\{A_t^n\}_{n \geq 1}$ is bounded. Define a new sequence B^n, $n = 0, 1, 2, \ldots$, as follows: $B^0 := A_t^0$ and B^n, $n = 1, 2, \ldots$, are defined inductively by

$$B^n := \frac{1}{1 - 2^{-n}} \left(B^{n-1} + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n} \right) \tag{23}$$

(notice that this is equivalent to (20) with B^n in place of A_t^n and $=$ in place of $\leq$). As $A_t^n \leq B^n$ for all n, it suffices to prove that B^n is bounded. If it is not, $B^N \geq 1$ for some N. By (23), $B^n \geq 1$ for all $n \geq N$. Therefore, again by (23),

$$B^n \leq B^{n-1} \frac{1}{1 - 2^{-n}} \left(1 + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n} \right), \quad n > N,$$

and the boundedness of the sequence B^n follows from $B^N < \infty$ and

$$\prod_{n=N+1}^{\infty} \frac{1}{1 - 2^{-n}} \left(1 + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n} \right) < \infty.$$

Thus, the sequence $\{A_t^n\}_{n \geq 1}$ is bounded, and from (22) it follows that:

$$\left| A_t^n - A_t^{n-1} \right| \leq O(n/2^n) \qquad \square$$

holds with lower probability at least $1 - \alpha$. Setting $\alpha \to 0$ finishes the proof.

Let Ω be equipped with the metric

$$\rho(\omega_1, \omega_2) := \sum_{d=1}^{\infty} 2^{-d} \sup_{t \in [0, 2^d]} (|\omega_1(t) - \omega_2(t)| \wedge 1) \tag{24}$$

(and the corresponding topology and Borel σ-algebra, the latter coinciding with $\mathcal{F}$). This makes it a complete and separable metric space. Therefore, Lemma 1 yields

Corollary 1. *It is almost certain that the sequence of functions $t \in [0, \infty) \mapsto A_t^n$ converges in Ω.*

For every $\omega \in \Omega$ the $\lim_{n \to \infty} A_t^n(\omega)$ (given that it exists) will be called *quadratic variation* of ω.

4 Riemannian quadratic variation

In the previous section we studied

$$A_t^n(\omega) := \sum_{k=0}^{\infty} \left(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\right)^2, \quad n = 0, 1, 2, \dots. \tag{25}$$

The function $A_t^n(\omega)$ is positive and it jumps at $t = T_k^n$ by the amount $\left(\omega(T_k^n) - \omega(T_{k-1}^n)\right)^2$, for $k = 0, 1, 2, \cdots$. Since $A_t^n(\omega)$ increases every time ω hits a *vertical* grid we may call $A_t(\omega)$ the *Lebesgue quadratic variation* of ω, given that it exists. Now, set the

$${}^{\mathcal{R}}A_t^n(\omega) := \sum_{k=1}^{\infty} \left(\omega(k2^{-n} \wedge t) - \omega((k-1)2^{-n} \wedge t)\right)^2, \quad n = 0, 1, 2, \dots. \tag{26}$$

This function resembles $A_t^n(\omega)$, but it increases at time points $t = k2^{-n}$, $k = 1, 2, \cdots$ by $(\omega(k2^{-n}) - \omega((k-1)2^{-n}))^2$. The function ${}^{\mathcal{R}}A_t^n$ accumulates every time ω hits a *horizontal* grid (see Figure 2). This justifies the following: if the $\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n(\omega)$ exists we call it the *Riemannian quadratic variation* of ω and denote it as ${}^{\mathcal{R}}A_t(\omega)$.

We would like to know whether ${}^{\mathcal{R}}A_t^n(\omega)$ converges to a limit.

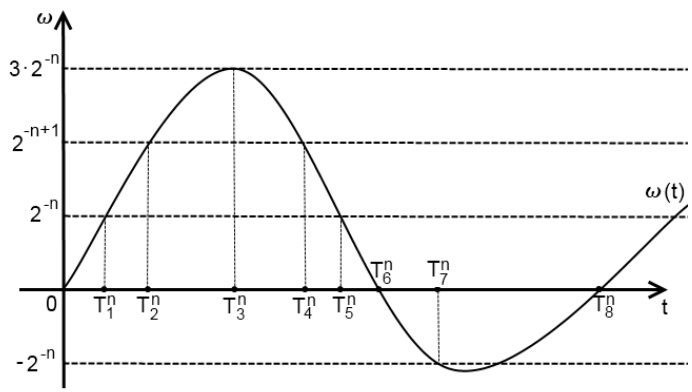
Lemma 2. *For any ω in the set*

$$\begin{aligned} \{\omega \in \Omega \mid \omega \textit{ is strictly monotone and } \exists c \in \mathbb{R}^+, \forall n \in \mathbb{N} : \\ \min_{k=1,\dots,\lceil T2^{-n}\rceil} \left|\omega((2k-1)2^{-n}) - \omega((2k-2)2^{n})\right| > c2^{-n/2}\} \end{aligned} \tag{27}$$

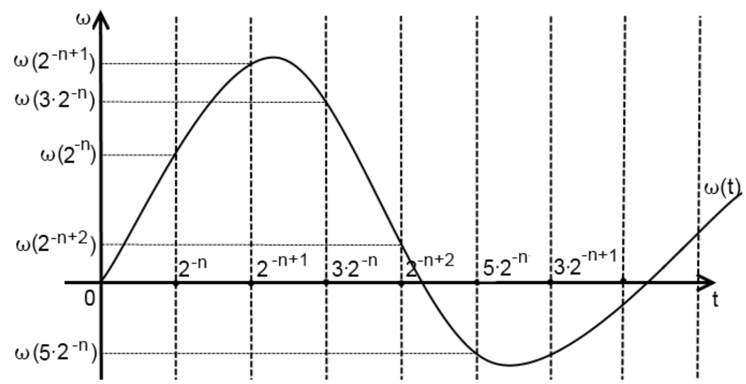
the $\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n(\omega)$ *does not exists for any $t \in [0, T]$.*

Proof. Fix $t \in [0, T]$ and $n \in \mathbb{N}$. Let $\tau := \lfloor t2^{-n+1} \rfloor$. Without the loss of generality, assume that $\lfloor t2^{-n+1} \rfloor = \lfloor t2^{-n} \rfloor$ (if this is not true we will have an additional term in (28), (29) and (30) which tends to 0 as $n \to \infty$). Consider

$$\begin{aligned} {}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^n(\omega) &= \sum_{k=1}^{\tau 2^{n-1}} \Big[\left(\omega(k2^{-n+1}) - \omega((k-1)2^{-n+1})\right)^2 \\ &- \left(\omega(k2^{-n+1}) - \omega((2k-1)2^{-n})\right)^2 - \left(\omega((2k-1)2^{-n}) - \omega((k-1)2^{-n+1})\right)^2\Big] \\ &= \sum_{k=1}^{\tau 2^{n-1}} 2\left(\omega((2k-1)2^{n}) - \omega((k-1)2^{-n+1})\right)\left(\omega(k2^{-n+1}) - \omega((2k-1)2^{-n})\right) \\ &= \sum_{k=1}^{\tau 2^{n-1}} 2\left(\omega((2k-1)2^{n}) - \omega((2k-2)2^{-n})\right)\left(\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\right) \end{aligned} \tag{28}$$



(a) Lebsgue



(b) Riemannian

Figure 2: Different constructions of quadratic variation.

The monotonicity of ω implies

$$\begin{aligned}
&\left|{}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^{n}(\omega)\right| \\
&= 2\sum_{k=1}^{\tau 2^{n-1}} \left|\big(\omega((2k-1)2^{n}) - \omega((2k-2)2^{-n})\big)\big(\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\big)\right| \\
&= 2\sum_{k=1}^{\tau 2^{n-1}} \left|\omega((2k-1)2^{n}) - \omega((2k-2)2^{-n})\right|\left|\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\right| \\
&\geq \tau 2^n \min_{k=1,\dots,\lceil T2^{-n}\rceil} \left|\omega((2k-1)2^{-n}) - \omega((2k-2)2^{n})\right|^2 .
\end{aligned} \tag{29}$$

From (27) we obtain

$$\left|{}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^{n}(\omega)\right| \geq \tau c. \tag{30}$$

From the definition of τ we have

$$\left|{}^{\mathcal{R}}A_t^{n-1}(\omega) - {}^{\mathcal{R}}A_t^n(\omega)\right| \geq \tau c. \tag{31}$$

This holds for any $n \in \mathbb{N}$. Hence $\{{}^{\mathcal{R}}A_t^n(\omega)\}_{n\geq 1}$ is not a Cauchy sequence, and, thus, does not have a limit. $\square$

Although, it is not true that $\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n(\omega)$ exists for all $\omega \in \Omega$, we still can define a class of continuous functions for which ${}^{\mathcal{R}}A_t^n(\omega)$ converges.

For any $\omega \in C[0,T]$, the function

$$m_\omega(\Delta) := \sup\{|\omega(t_1) - \omega(t_2)| : (t_1, t_2 \in [0,T]) \wedge |t_1 - t_2| \leq \Delta\} \tag{32}$$

is called the *modulus of continuity of* ω.

Lemma 3. *For any $\omega \in C[0,T]$ such that*

$$m_\omega(\Delta) = O(\Delta) \text{ as } \Delta \to 0 \tag{33}$$

${}^{\mathcal{R}}A_t(\omega)$ exists for any $t \in [0,T]$.

Proof. Fix $t \in [0,T]$ and $n \in \mathbb{N}$. Again, let $\tau := \lfloor t2^{-n+1} \rfloor$ and without the loss of generality, suppose that $\lfloor t2^{-n+1} \rfloor = \lfloor t2^{-n} \rfloor$ (if not, then there will have additional an term in (34), (35) and (36) which tends to 0 as $n \to \infty$).

Recall from (28) that

$$\begin{aligned}&{}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^n(\omega)\\ &= \sum_{k=1}^{\tau 2^{n-1}} 2\left(\omega((2k-1)2^n) - \omega((2k-2)2^{-n})\right)\left(\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\right)\end{aligned} \tag{34}$$

From (33) we have

$$\begin{aligned}&{}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^n(\omega)\\ &\geq -\sum_{k=1}^{\tau 2^{n-1}} 2\left|\omega((2k-1)2^{-n}) - \omega((2k-2)2^{-n})\right|\left|\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\right|\\ &\geq -\tau 2^n O(2^{-n}) O(2^{-n}) = -O(2^{-n}).\end{aligned} \tag{35}$$

Similarly, we can show that

$$\begin{aligned}&{}^{\mathcal{R}}A_\tau^{n-1}(\omega) - {}^{\mathcal{R}}A_\tau^n(\omega)\\ &\leq \sum_{k=1}^{\tau 2^{n-1}} 2\left|\omega((2k-1)2^{-n}) - \omega((2k-2)2^{-n})\right|\left|\omega(2k2^{-n}) - \omega((2k-1)2^{-n})\right|\\ &\leq O(2^{-n}).\end{aligned} \tag{36}$$

From the way we defined τ and from (35) and (36), we have

$$|{}^{\mathcal{R}}A_t^n(\omega) - {}^{\mathcal{R}}A_t^{n-1}(\omega)| \le O(2^{-n}). \tag{37}$$

Therefore, $\{{}^{\mathcal{R}}A_t^n(\omega)\}_{n\ge1}$ is a Cauchy sequence and, thus, has a limit. $\square$

Lemma 4. *Under the conditions of the previous lemma*

$$\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n = \lim_{n\to\infty} A_t^n$$

almost surely.

Proof. Fix $t \in [0,T]$ and fix $n \in \mathbb{N}$. Let ω be such that A_t exists (it happens almost surely) and let K be the largest natural number such that $T_K^n \le t$.

Define

$$q_n := \min\{z \in \mathbb{N} \mid \forall k \in \{0,1,\dots,K\}\ \exists l \in \mathbb{N} : T_k^n \le l2^{-z} \le T_{k-1}^n\} \tag{38}$$

From this definition it follows that

$$K \le t2^{q_n}. \tag{39}$$

Since $\{{}^{\mathcal{R}}A_t^{q_n}\}_{n\ge1}$ is a subsequence of $\{{}^{\mathcal{R}}A_t^n\}_{n\ge1}$ we have

$$\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n = \lim_{n\to\infty} {}^{\mathcal{R}}A_t^{q_n} \tag{40}$$

For each $k = 0,\dots,K$, let $\tau_k^n \in \{l2^{-q_n} \mid l \in \mathbb{N}\}$ be such that

$$T_k^n - \tau_k^n = \min_{t\in\{l2^{-q_n}|l\in\mathbb{N}\}} \{T_k^n - t \mid T_k^n - t \ge 0\}$$

and

$$e_k^n := T_k^n - \tau_k^n.$$

Note that $e_k^n > 0$ and

$$\omega(\tau_k^n) = \omega(T_k^n - e_k^n) = \omega(T_k^n) + \omega(T_k^n - e_k^n) - \omega(T_k^n).$$

By the statement of the lemma

$$|\omega(T_k^n - e_k^n) - \omega(T_k^n)| = O(e_k^n) \le O(2^{-q_n}).$$

Hence

$$\omega(T_k^n) - O(2^{-q_n}) \le \omega(\tau_k^n) \le \omega(T_k^n) + O(2^{-q_n}). \tag{41}$$

Consider

$$\begin{aligned} A_t^n - {}^{\mathcal{R}}A_t^{q_n} &= \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2 - \sum_{k=1}^{\lfloor t2^{q_n}\rfloor} \big(\omega(k2^{-q_n}) - \omega((k-1)2^{-q_n})\big)^2 \\ &\le \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2 - \sum_{k=1}^{K} \big(\omega(\tau_k^n) - \omega(\tau_{k-1}^n)\big)^2 \end{aligned} \tag{42}$$

It follows from (41) that

$$\begin{aligned}
&A_t^n - {}^{\mathcal{R}}A_t^{q_n} \\
&\quad\le \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2 - \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n) + 2O(2^{-q_n})\big)^2 \\
&\qquad = \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2 - \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2 \\
&\qquad + 2O(2^{-2q_n+2}) \sum_{k=1}^{K} \big(\omega(T_k^n) - \omega(T_{k-1}^n)\big) + KO(2^{-4q_n+4}) \\
&\qquad\qquad = O(2^{-2q_n+3})\,(\omega(T_K^n) - \omega(T_0^n)) + KO(2^{-4q_n+4}). \quad (43)
\end{aligned}$$

By (39) we get

$$A_t^n - {}^{\mathcal{R}}A_t^{q_n} \le O(2^{-2q_n+3})\,(\omega(T_K^n) - \omega(T_0^n)) + tO(2^{-3q_n+4}). \quad (44)$$

Now, let

$$v_n := \min\{z \in \mathbb{N} \mid \forall k \in \{1, \dots, \lfloor t2^n \rfloor\}\ \exists l \in \mathbb{N} : k2^{-n} \le T_l^z \le (k-1)2^{-n}\} \quad (45)$$

and

$$K_{v_n} := \frac{A_T^{v_n} - A_0^{v_n}}{2^{-2v_n}}.$$

Obviously

$$\lim_{n\to\infty} A_t^n = \lim_{n\to\infty} A_t^{v_n}. \quad (46)$$

For every $k = 1, \dots, \lfloor t2^n \rfloor$, let $T_{p_k}^{v_n}$ be such that

$$k2^{-n} - T_{p_k}^{v_n} = \min_{l=0,\dots,K_{v_n}} \{k2^{-n} - T_l^{v_n} \mid k2^{-n} - T_l^{v_n} \ge 0\}.$$

Also, define

$$d_k^n := k2^{-n} - T_{p_k}^{v_n}.$$

Then

$$\omega(T_{p_k}^{v_n}) = \omega(k2^{-n} - d_k^n)$$

and by slightly different reasoning we used to deduce (41) one can show

$$\omega(k2^{-n}) - O(2^{-n}) \le \omega(T_{p_k}^{v_n}) \le \omega(k2^{-n}) + O(2^{-n}). \quad (47)$$

Consider

$$\begin{aligned}
{}^{\mathcal{R}}A_t^n - A_t^{v_n} &= \sum_{k=1}^{\lfloor t2^n \rfloor} \big(\omega(k2^{-n}) - \omega((k-1)2^{-n})\big)^2 - \sum_{k=1}^{K_{v_n}} \big(\omega(T_k^{v_n}) - \omega(T_{k-1}^{v_n})\big)^2 \\
&\le \sum_{k=1}^{\lfloor t2^n \rfloor} \big(\omega(k2^{-n}) - \omega((k-1)2^{-n})\big)^2 - \sum_{k=1}^{\lfloor t2^n \rfloor} \Big(\omega(T_{p_k}^{v_n}) - \omega(T_{p_{k-1}}^{v_n})\Big)^2 \quad (48)
\end{aligned}$$

By (47) we have

$$
\begin{aligned}
{}^{\mathcal{R}}A_t^n - A_t^{v_n} \le \sum_{k=1}^{\lfloor t2^n \rfloor} &\big(\omega(k2^{-n}) - \omega((k-1)2^{-n})\big)^2 \\
&- \sum_{k=1}^{\lfloor t2^n \rfloor} \big(\omega(k2^{-n}) - \omega((k-1)2^{-n}) + 2O(2^{-n})\big)^2 \\
= O(2^{-2n+3}) &\sum_{k=1}^{\lfloor t2^n \rfloor} \big(\omega(k2^{-n}) - \omega((k-1)2^{-n})\big) + (\lfloor t2^n \rfloor - 1)O(2^{-4n+4}) \\
&\le O(2^{-2n+3}) \big(\omega(\lfloor t2^n \rfloor 2^{-n}) - \omega(0)\big) + tO(2^{-3n+4}) \qquad (49)
\end{aligned}
$$

From the fact that ω is bounded on $[0, T]$ and from (40), (44), (46) and (49) we obtain

$$\lim_{n\to\infty} {}^{\mathcal{R}}A_t^n = \lim_{n\to\infty} A_t^n \tag{50}$$

□

5 Relative quadratic variation

Let Ω^+ be the set of all continuous functions $\omega : [0, \infty) \to (0, \infty)$ (we may also write $C^+[0, T]$ instead of Ω^+). In this section we will consider only $\omega \in \Omega^+$.

Define

$$R_t^n(\omega) := \sum_{k=0}^{\infty} \frac{\big(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\big)^2}{\omega(T_{k-1}^n \wedge t)^2}, \quad n = 0, 1, 2, \dots. \tag{51}$$

If $\lim_{n\to\infty} R_t^n(\omega)$ exists, we call it a *relative quadratic variation* of ω and denote as $R_t(\omega)$.

Lemma 5. *For each $T > 0$, it is almost certain that $t \in [0, T] \mapsto R_t^n$ is a Cauchy sequence of functions in $C^+[0, T]$.*

Proof. Fix $T > 0$, fix $\omega \in C^+[0, T]$ and fix $n \in \{1, 2, \dots\}$. Assume, for simplicity, that $T_0^n = T_0^{n-1}$ (the reasoning for the other case is similar). Let Z_K be the number of times R_t^{n-1} increased over the time interval $[0, T_K^n]$ less one. Since A_t^{n-1} increases at the same time R_t^{n-1} does, Z_K is also equal to the number of times $A_t^{n-1}(\omega)$ increased on $[0, T_K^n]$ less one. Therefore,

$$Z_K = \frac{A_{T_K^n}^{n-1} - A_0^{n-1}}{2^{-2n+2}}. \tag{52}$$

For any $K = 1, 2, \dots$

$$
\begin{aligned}
R_{T_K^n}^n - R_{T_K^n}^{n-1} &= \sum_{k=1}^{K} \frac{\big(\omega(T_k^n) - \omega(T_{k-1}^n)\big)^2}{\omega(T_{k-1}^n)^2} - \sum_{k=1}^{Z_K} \frac{\big(\omega(T_k^{n-1}) - \omega(T_{k-1}^{n-1})\big)^2}{\omega(T_{k-1}^{n-1})^2} \\
&= 2^{-2n} \sum_{k=1}^{K} \frac{1}{\omega(T_{k-1}^n)^2} - 2^{-2n+2} \sum_{k=1}^{Z_K} \frac{1}{\omega(T_{k-1}^{n-1})^2}.
\end{aligned}
\tag{53}
$$

Define

$$\sigma^+_{n,K} := \frac{1}{\left(\omega(T_0^n) - K2^{-n}\right)^2} - \frac{1}{\omega(T_0^n)^2}$$
$$= \frac{2^{-n+1}K\omega(T_0^n) - K^2 2^{-2n}}{\omega(T_0^n)^2 \left(\omega(T_0^n) - K2^{-n}\right)^2}.$$

and

$$\sigma^-_{n,K} := \frac{1}{\left(\omega(T_0^n) + K2^{-n}\right)^2} - \frac{1}{\omega(T_0^n)^2}$$
$$= -\frac{2^{-n+1}K\omega(T_0^n) + K^2 2^{-2n}}{\omega(T_0^n)^2 \left(\omega(T_0^n) + K2^{-n}\right)^2}.$$

Since, the sequence $\{A_t^n\}_{n\geq 1}$ is bounded (see the proof of Lemma 1), from (17) and (52) we may infer that $K = O(2^{2n})$ and $Z_K = O(2^{2n-2})$. Now, it is easy to see that

$$\sigma^+_{n,K}, \sigma^-_{n,K}, \sigma^+_{n-1,Z_K}, \sigma^-_{n-1,Z_K} \to -\frac{1}{\omega(T_0^n)^2}, \text{ as } n \to \infty. \tag{54}$$

For any $i = 1, \ldots, K$ and any $j = 1, \ldots, Z_K$ it is true that

$$\frac{1}{\left(\omega(T_0^n) + K2^{-n}\right)^2} \leq \frac{1}{\omega(T_i^n)^2} \leq \frac{1}{\left(\omega(T_0^n) - K2^{-n}\right)^2}. \tag{55}$$

and

$$\frac{1}{\left(\omega(T_0^{n-1}) + Z_K 2^{-n+1}\right)^2} \leq \frac{1}{\omega(T_j^{n-1})^2} \leq \frac{1}{\left(\omega(T_0^{n-1}) - Z_K 2^{-n+1}\right)^2}. \tag{56}$$

Expressions (55) and (56) are equivalent to

$$\frac{1}{\omega(T_0^n)^2} + \sigma^-_{n,K} \leq \frac{1}{\omega(T_i^n)^2} \leq \frac{1}{\omega(T_0^n)^2} + \sigma^+_{n,K}. \tag{57}$$

and

$$\frac{1}{\omega(T_0^{n-1})^2} + \sigma^-_{n-1,Z_K} \leq \frac{1}{\omega(T_j^{n-1})^2} \leq \frac{1}{\omega(T_0^{n-1})^2} + \sigma^+_{n-1,Z_K}. \tag{58}$$

respectively.

It follows from (53), (57) and (58) that

$$
\begin{aligned}
R^n_{T^n_K} - R^{n-1}_{T^n_K} &\le 2^{-2n} \sum_{k=1}^{K} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right) - 2^{-2n+2} \sum_{k=1}^{Z_K} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n-1,Z_K} \right) \\
&= 2^{-2n} K \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right) - 2^{-2n+2} Z_K \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n-1,Z_K} \right) \\
&= \left(A^n_{T^n_K} - A^n_{T^n_0} \right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right) - \left(A^{n-1}_{T^n_K} - A^{n-1}_{T^n_0} \right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n-1,Z_K} \right) \\
&= \frac{A^n_{T^n_K} - A^{n-1}_{T^n_K}}{\omega(T^n_0)^2} + A^n_{T^n_K} \sigma^+_{n,K} - A^{n-1}_{T^n_K} \sigma^-_{n-1,Z_K} + 2^{-2n+2} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n-1,Z_K} \right).
\end{aligned}
\tag{59}
$$

On the other hand,

$$
\begin{aligned}
R^n_{T^n_K} - R^{n-1}_{T^n_K} &\ge 2^{-2n} \sum_{k=1}^{K} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right) - 2^{-2n+2} \sum_{k=1}^{Z_K} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n-1,Z_K} \right) \\
&= \left(A^n_{T^n_K} - A^n_{T^n_0} \right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right) - \left(A^{n-1}_{T^n_K} - A^{n-1}_{T^n_0} \right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n-1,Z_K} \right) \\
&= \frac{A^n_{T^n_K} - A^{n-1}_{T^n_K}}{\omega(T^n_0)^2} + A^n_{T^n_K} \sigma^-_{n,K} - A^{n-1}_{T^n_K} \sigma^+_{n-1,Z_K} - 2^{-2n} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right).
\end{aligned}
\tag{60}
$$

Inequalities (59) and (60) are true for any $K = 0, 1, \ldots$. Choosing the largest K such that $T^n_K \le t$ (the case $T^{n-1}_0 \ge t$ is trivial) yields

$$
R^n_t - R^{n-1}_t \le \frac{A^n_t - A^{n-1}_t}{\omega(T^n_0)^2} + A^n_t \sigma^+_{n,K} - A^{n-1}_t \sigma^-_{n-1,Z_K} + 2^{-2n+2} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n-1,Z_K} \right). \tag{61}
$$

and

$$
R^n_t - R^{n-1}_t \ge \frac{A^n_t - A^{n-1}_t}{\omega(T^n_0)^2} + A^n_t \sigma^-_{n,K} - A^{n-1}_t \sigma^+_{n-1,Z_K} + 2^{-2n} \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right). \tag{62}
$$

The claim of this Lemma follows from setting $n \to \infty$ in (54), (61), from (62) and from Lemma 1. □

Recall, that Ω is equipped with metric defined by (24). Therefore,

Corollary 2. *It is almost certain that the sequence of functions* $t \in [0, \infty) \mapsto R^n_t$ *converges in* Ω.

The following lemma relates relative quadratic variation to absolute one.

Lemma 6. *For any* $t \in [0, \infty)$, *it is almost certain that*

$$\lim_{n\to\infty} R^n_t = \int_0^t \frac{1}{\omega(s)^2} dA_s. \tag{63}$$

Proof. Suppose that ω is such that $R_t(\omega)$ exists (this is true for any $\omega \in C^+[0, T]$ almost surely). Consider

$$\begin{aligned} R^n_{T^n_K} - \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} &= \sum_{k=1}^{K} \frac{\left(\omega(T^n_k) - \omega(T^n_{k-1})\right)^2}{\omega(T^n_{k-1})^2} - \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} \\ &= 2^{-2n} \sum_{k=1}^{K} \frac{1}{\omega(T^n_{k-1})^2} - \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2}. \end{aligned} \tag{64}$$

From (57) and (58) we deduce

$$\begin{aligned} & R^n_{T^n_K} - \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} \\ & \quad \le 2^{-2n} K \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right) - \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right) \sum_{k=1}^{K} \left(A_{T^n_k} - A_{T^n_{k-1}}\right) \\ & \quad = \left(A^n_{T^n_K} - A^n_{T^n_0}\right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right) - \left(A_{T^n_K} - A_{T^n_0}\right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right). \end{aligned} \tag{65}$$

In the same way we can show that

$$\begin{aligned} & R^n_{T^n_K} - \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} \\ & \quad \ge \left(A^n_{T^n_K} - A^n_{T^n_0}\right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^-_{n,K} \right) - \left(A_{T^n_K} - A_{T^n_0}\right) \left(\frac{1}{\omega(T^n_0)^2} + \sigma^+_{n,K} \right). \end{aligned} \tag{66}$$

The inequalities (65) and (66) together with (54) yield

$$\lim_{n\to\infty} R^n_{T^n_K} = \lim_{n\to\infty} \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} \tag{67}$$

Choose the largest K such that $T^n_K \le t$. From (67) and the definition of Lebesgue-Stieljes integral we get

$$R_t = \lim_{n\to\infty} R^n_t = \lim_{n\to\infty} \sum_{k=1}^{K} \frac{\left(A_{T^n_k} - A_{T^n_{k-1}}\right)}{\omega(T^n_{k-1})^2} = \int_0^t \frac{1}{\omega(s)^2} dA_s. \qquad (68)$$

□

6 Conclusion and further research

This work further develops the concept of quadratic variation in game-theoretic probability as defined by Vovk in [5], [6] and [8]. In particular, we show that quadratic variation defined in Riemannian sense exists under certain conditions and coincides almost surely with the Lebesgue quadratic variation. We also show that the relative quadratic variation exists and is equal to Lebesgue-Stiltjes integral defined via its absolute counterpart. In further research using the results of [9], [8] and this paper we would like to develop the game-theoretic capital asset pricing model in continuous time. Specifically, we would like to prove the analogies to (3) and (4) in this setting.

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